# MATHEMATICAL FORMULATION OF THE KINEMATIC EQUATIONS FOR THE CONTROL OF THE ROBOT SYSTEM WITH APPLICATION FOR THE MACHINING CONICAL SURFACES

**STEFAN GASPAR, JAN PASKO**

Technical University of Kosice, Faculty of Manufacturing Technologies with a seat in Presov, Department of Design Technical Systems, Presov, Slovak Republic

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**e-mail: stefan.gaspar@tuke.sk**

With the performance of various technological operations (welding, machining, etc.) it is necessary that the end member of a robot moves along the defined path and, at the same time, the tool fulfils certain criteria in relation to the machined surface. The aim of the article is to apply theoretical knowledge to derivation of kinematic equations of a robotic mechanism control, i.e. to the solution of an inverse problem. The substance of the problem consists it the fact that based on the required position and orientation of the tool, it is necessary to determine the values of the generalized coordinates in individual joints of a robot. The solving method of the presented problem is illustrated by the specific example, i.e. the movement of the end member of a robot with conical surfaces machining.

**KEYWORDS**

robotic mechanism, conical surface, orientation of a robot end element, transform matrix, inverse problem

## 1 THE DESIGN OF THE ROBOTIC MECHANISM

The designed robotic mechanism is presented in Fig. 1. The mechanism has six degrees of freedom, a simple open kinematic structure with rotary kinematic couples only. We have chosen this design of the mechanism because of the aim of its application and optimality with the regard to the analysis and the solution of the so-called inverse problem. On the other hand the position and the orientation of the end member (tool) is given by six parameters and thus the mechanism must have minimally six degrees of the freedom.

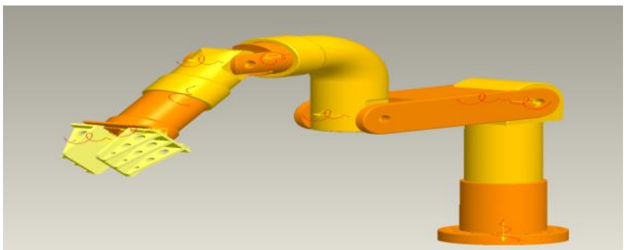

**Figure 1.** Mechanism of a robot

## 2 GEOMETRICAL DESCRIPTION OF THE MECHANISM

In this part we take interest in the geometrical description of the considered mechanism. We denote its edges successively by. Then the location and the orientation of the end member are uniquely given by six so-called generalized coordinates. Those are the angles of rotations the meaning of which is obvious from Fig. 2. In order to make the analytical description of the mechanism, we express the related points and vectors by the coordinates in the orthonormal anticlockwise oriented systems related to the members of the mechanism as it is shown in Fig. 2. Besides these coordinate systems we consider the system connected to the base (by a workpiece) and connected to the tool. In order to simplify the following notations, we denote and. In the sequel we denote the coordinates of the point by and the coordinates of the vector by in the coordinate system. Between the coordinates of the points in the individual coordinate systems the following transform equations will hold. [Modrak 2002, Pasko 2008]

$$X^{[j]} = T_{j,i} X^{[i]} + P_i^{[j]} \tag{1}$$

$$T_{j,i} = \left[e_i^{[j]}, f_i^{[j]}, g_i^{[j]}\right] = \begin{bmatrix} \vec{e}_i \cdot \vec{e}_j & \vec{f}_i \cdot \vec{e}_j & \vec{g}_i \cdot \vec{e}_j \\ \vec{e}_i \cdot \vec{f}_j & \vec{f}_i \cdot \vec{f}_j & \vec{g}_i \cdot \vec{f}_j \\ \vec{e}_i \cdot \vec{g}_j & \vec{f}_i \cdot \vec{g}_j & \vec{g}_i \cdot \vec{g}_j \end{bmatrix}, i, j = -1, 0, 1, \cdots, 6, 7 \tag{2}$$

If we apply the notation of the so-called extended coordinates and transform matrices,

$$\overline{X}^{[i]} = \begin{bmatrix} X^{[i]} \\ 1 \end{bmatrix} = \begin{bmatrix} \overrightarrow{P_i X} \cdot \vec{e}_i \\ \overrightarrow{P_i X} \cdot \vec{f}_i \\ \overrightarrow{P_i X} \cdot \vec{g}_i \\ 1 \end{bmatrix}, \overline{T}_{j,i} = \begin{bmatrix} T_{j,i} & P_i^{[j]} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \vec{e}_i \cdot \vec{e}_j & \vec{f}_i \cdot \vec{e}_j & \vec{g}_i \cdot \vec{e}_j & \overrightarrow{P_i P_j} \cdot \vec{e}_i \\ \vec{e}_i \cdot \vec{f}_j & \vec{f}_i \cdot \vec{f}_j & \vec{g}_i \cdot \vec{f}_j & \overrightarrow{P_i P_j} \cdot \vec{f}_i \\ \vec{e}_i \cdot \vec{g}_j & \vec{f}_i \cdot \vec{g}_j & \vec{g}_i \cdot \vec{g}_j & \overrightarrow{P_i P_j} \cdot \vec{g}_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

we can rewrite the transform equations (1) in the compact form

$$\overline{X}^{[j]} = \overline{T}_{j,i} \overline{X}^{[i]}, i, j = -1, 0, 1, \cdots, 6, 7 \tag{4}$$

Since the repeated transition from one coordinate system to another one is given by the product of the related extended transform matrices, it is easy to verify that the following relations hold

$$\overline{T}_{j,i} = \overline{T}_{j,j+1} \overline{T}_{j+1,j+2} \cdots \overline{T}_{i-2,i-1} \overline{T}_{i-1,i}, \quad \overline{T}_{i,j} = \left(\overline{T}_{j,i}\right)^{-1}, -1 \le j < i \le 7 \tag{5}$$

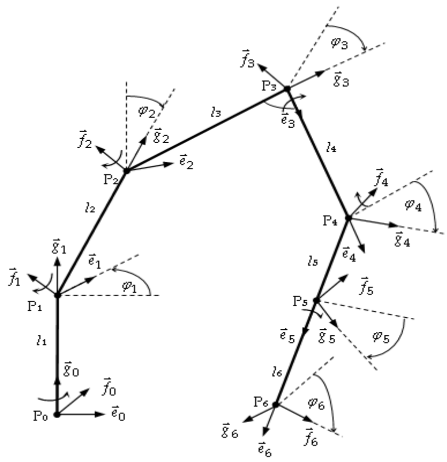


**Figure 2.** Coordinate systems and the generalized coordinates

Finally, if we denote

$$c_i = \cos\varphi_i, s_i = \sin\varphi_i,$$

we get the relations for the basic transform matrices and for their inverse ones [Xiang 2004, Zhifei 2014 ]

$$\overline{T}_{0,1} = \left[\begin{array}{ccc:c} c_1 & s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & l_1 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right], \quad \overline{T}_{1,2} = \left[\begin{array}{ccc:c} c_2 & 0 & s_2 & l_2 s_2 \\ 0 & 1 & 0 & 0 \\ -s_2 & 0 & c_2 & l_2 c_2 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right],$$

$$\overline{T}_{2,3} = \left[\begin{array}{ccc:c} c_3 & 0 & s_3 & l_3 s_3 \\ 0 & 1 & 0 & 0 \\ -s_3 & 0 & c_3 & l_3 c_3 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right], \quad \overline{T}_{3,4} = \left[\begin{array}{ccc:c} 1 & 0 & 0 & l_4 \\ 0 & c_4 & -s_4 & 0 \\ 0 & s_4 & c_4 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right],$$

$$\overline{T}_{4,5} = \left[\begin{array}{ccc:c} c_5 & 0 & s_5 & l_5 c_5 \\ 0 & 1 & 0 & 0 \\ -s_5 & 0 & c_5 & -l_5 s_5 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right], \quad \overline{T}_{5,6} = \left[\begin{array}{ccc:c} 1 & 0 & 0 & l_6 \\ 0 & c_6 & -s_6 & 0 \\ 0 & s_6 & c_6 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right], \qquad (6)$$

$$\left(\overline{T}_{0,1}\right)^{-1} = \left[\begin{array}{ccc:c} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & -l_1 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right], \quad \left(\overline{T}_{1,2}\right)^{-1} = \left[\begin{array}{ccc:c} c_2 & 0 & -s_2 & 0 \\ 0 & 1 & 0 & 0 \\ s_2 & 0 & c_2 & -l_2 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right],$$

$$\left(\overline{T}_{2,3}\right)^{-1} = \left[\begin{array}{ccc:c} c_3 & 0 & -s_3 & 0 \\ 0 & 1 & 0 & 0 \\ s_3 & 0 & c_3 & -l_3 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right], \quad \left(\overline{T}_{3,4}\right)^{-1} = \left[\begin{array}{ccc:c} 1 & 0 & 0 & -l_4 \\ 0 & c_4 & s_4 & 0 \\ 0 & -s_4 & c_4 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right],$$

$$\left(\overline{T}_{4,5}\right)^{-1} = \left[\begin{array}{ccc:c} c_5 & 0 & -s_5 & -l_5 \\ 0 & 1 & 0 & 0 \\ s_5 & 0 & c_5 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right], \quad \left(\overline{T}_{5,6}\right)^{-1} = \left[\begin{array}{ccc:c} 1 & 0 & 0 & -l_6 \\ 0 & c_6 & s_6 & 0 \\ 0 & -s_6 & c_6 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] \qquad (7)$$

## 3 THE INVERSE PROBLEM AND ITS SOLUTION

Let us suppose that there are given the lengths of the individual members of the mechanism and the extended transform matrices [Brat 1981]

- $\overline{Z} = \overline{T}_{-1,0}$ - the given location and orientation of the mechanism with respect to the base,
- $\overline{W} = \overline{T}_{6,7}$ - the given location and orientation of the tool with respect to the end member,
- $\overline{T} = \overline{T}(t)$ the required location and orientation of the tool with respect to the base at the given time instant. [Pasko 2008]

The aim is to determine the values of the generalized coordinates $\varphi_i = \varphi_i(t), i = 1, 2, \cdots, 6$ so that the transform matrix $\overline{T}_{-1,7} = \overline{T}_{-1,7}(\varphi_1(t), \varphi_2(t), \cdots, \varphi_6(t))$ holds the equality

$$\overline{T}_{-1,7} = \overline{T} \qquad (8)$$

This equality leads to twelve nonlinear equations with six unknowns. The solution to them using some iterative method may cause some problems with the inexactness, non-uniqueness, and with the excluding the superfluous equations. Therefore we apply the geometrical approach to the solution of this inverse problem by the following steps. [Modrak 2002, Pasko 2008]

1. By (7) and (5) it is possible to calculate the transform matrix

$$\overline{T}_{0,6} = \overline{Z}^{-1}\overline{T}\overline{W}^{-1} = \left[\begin{array}{ccc:c} e_6^{[0]} & f_6^{[0]} & g_6^{[0]} & P_6^{[0]} \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]$$

2. Based on Fig. 2 it is obvious that $P_5 = P_6 - l_6\vec{e}_6, \vec{e}_5 = \vec{e}_6$ a $P_4 = P_5 - l_5\vec{e}_5$ which at the coordinate system $S_0$ gives the relation

$$P_4^{[0]} = \begin{bmatrix} x_4^{[0]} \\ y_4^{[0]} \\ z_4^{[0]} \end{bmatrix} = P_6^{[0]} - (l_5 + l_6)e_6^{[0]}$$

3. Let be the orthogonal projection of the point to the plain, then

$$Q_4^{[0]} = \begin{bmatrix} x_4^{[0]} \\ y_4^{[0]} \\ 0 \end{bmatrix}$$

and the angle $\varphi_1$ is the oriented one between the vectors $\vec{e}_0$ and $\overrightarrow{P_0Q_4}$ Hence (see Fig. 3)

$$\varphi_1 = \left(\operatorname{sgn} y_4^{[0]}\right)\arccos\frac{x_4^{[0]}}{d_4} \text{ if } d_4 = \sqrt{\left(x_4^{[0]}\right)^2 + \left(y_4^{[0]}\right)^2} \neq 0 \qquad (9)$$

and the angle $\varphi_1$ can be chosen arbitrarily if $d_4 = 0$ . By (6) and (7) we know the transform matrices $\overline{T}_{0,1}$ and $\left(\overline{T}_{0,1}\right)^{-1}$ and $\left(\overline{T}_{0,1}\right)^{-1}$ and by (5) also

$$\overline{T}_{1,6} = \left(\overline{T}_{0,1}\right)^{-1}\overline{T}_{0,6} = \left[\begin{array}{ccc:c} e_6^{[1]} & f_6^{[1]} & g_6^{[1]} & P_6^{[1]} \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]. \text{ [2, 9]}$$

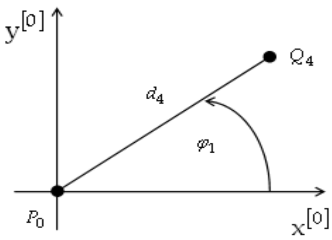


**Figure 3.** Determination of the angle ϕ1

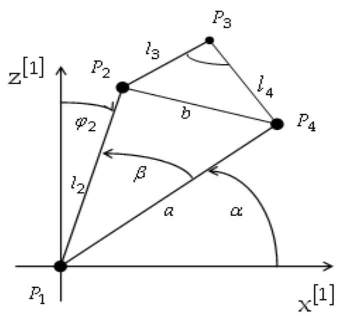


**Figure 4.** Determination of the angle ϕ2

4. By the analogous way as in the step 2, with the coordinate system $S_1$ we get

$$P_4^{[1]} = \begin{bmatrix} x_4^{[1]} \\ 0 \\ z_4^{[1]} \end{bmatrix} = P_6^{[1]} - (l_5 + l_6)e_6^{[1]}$$

Let us denote $a = \sqrt{\left(x_4^{[1]}\right)^2 + \left(z_4^{[1]}\right)^2}$, $b = \sqrt{l_3^2 + l_4^2}$. If $a = 0$ and $l_2 = b$, the angle $\varphi_2$ can be chosen arbitrarily. If $a = 0$ and $l_2 \neq b$, the problem has no solution. If $a \neq 0$, then based on fig. 4 as well as the cosine theorem.

$$\alpha = \mathrm{sgn}\left(z_4^{[1]}\right)\arccos\frac{x_4^{[1]}}{a}, \quad \beta = \arccos\frac{b^2 - a^2 - l_2^2}{2al_2},$$

$$\varphi_2 = 90^0 - (\alpha + \beta) \tag{10}$$

By (6) and (7) we know the transform matrices   and by (5) also

$$\overline{T}_{2,6} = \left(\overline{T}_{1,2}\right)^{-1}\overline{T}_{1,6} = \left[\begin{array}{ccc|c} e_6^{[2]} & f_6^{[2]} & g_6^{[2]} & P_6^{[2]} \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$

5. By the analogous way as in the preceding it is [Pasko 2008]

$$P_4^{[2]} = \begin{bmatrix} x_4^{[2]} \\ 0 \\ z_4^{[2]} \end{bmatrix} = P_6^{[2]} - (l_5 + l_6)e_6^{[2]}.$$

And based on Fig. 5. [Modrak 2002]

$$\gamma = \mathrm{sgn}\left(z_4^{[2]}\right)\arccos\frac{x_4^{[2]}}{b}, \quad \delta = \arccos\frac{l_3}{b}, \quad \varphi_3 = 90^0 - (\gamma + \delta) \tag{11}$$

Hence, we know the transform matrices $\overline{T}_{2,3}$ and $\left(\overline{T}_{2,3}\right)^{-1}$,

and $\overline{T}_{3,6} = \left(\overline{T}_{2,3}\right)^{-1}\overline{T}_{2,6} = \left[\begin{array}{ccc|c} e_6^{[3]} & f_6^{[3]} & g_6^{[3]} & P_6^{[3]} \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$

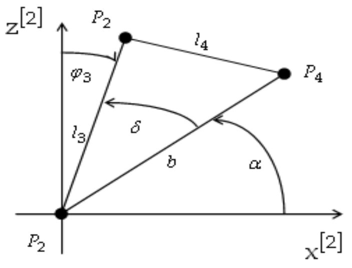


**Figure 5.** Determination of the angle $\phi_1$

6. From Fig. 2 we can see that $\varphi_5$ is the angle between the vectors $\vec{e}_3$ and $\vec{e}_6$. Hence

$$\varphi_5 = \arccos\left(\vec{e}_3 \cdot \vec{e}_6\right) = \arccos\left(e_3^{[3]} \cdot e_6^{[3]}\right) \tag{12}$$

7. If $\varphi_5 = 0^0$ or $\varphi_5 = 180^0$, the angle $\varphi_4$ can be chosen arbitrarily. Then

$\vec{f}_4 = \vec{f}_5 = \cos\varphi_4 \vec{f}_3 + \sin\varphi_4 \vec{g}_3$, i.e

$$f_4^{[3]} = f_5^{[3]} = \cos\varphi_4 f_3^{[3]} + \sin\varphi_4 g_3^{[3]} = \begin{bmatrix} 0 \\ \cos\varphi_4 \\ \sin\varphi_4 \end{bmatrix}.$$

8. If $\varphi_5 \neq 0^0$ and $\varphi_5 \neq 180^0$ from figure 2 we can see that the vector $\vec{f}_4 = \vec{f}_5$ is orthogonal to the vectors $\vec{e}_3$ and $\vec{e}_6$, and we can put

$$\vec{f}_4 = \vec{f}_5 = \frac{\vec{e}_3 \times \vec{e}_6}{\left|\vec{e}_3 \times \vec{e}_6\right|} = \frac{e_3^{[3]} \times e_6^{[3]}}{\left|e_3^{[3]} \times e_6^{[3]}\right|}.$$

9. Finally, we can complete the coordinate systems $S_4$ and $S_5$ to anticlockwise oriented ones by the vectors

$\vec{e}_4 = \vec{e}_3, \vec{g}_4 = \vec{e}_4 \times \vec{f}_4, \vec{e}_5 = \vec{e}_6, \vec{g}_5 = \vec{e}_5 \times \vec{f}_5$, i.e

$$e_4^{[3]} = e_3^{[3]}, g_4^{[3]} = e_4^{[3]} \times f_4^{[3]}, e_5^{[3]} = e_6^{[3]}, g_5^{[3]} = e_5^{[3]} \times f_5^{[3]}$$

and for the rest of the angles we get [1, 2]

$$\varphi_4 = \arccos\left(\vec{g}_3 \cdot \vec{g}_4\right) = \arccos\left(g_3^{[3]} \cdot g_4^{[3]}\right),$$
$$\varphi_6 = \arccos\left(\vec{g}_5 \cdot \vec{g}_6\right) = \arccos\left(g_5^{[3]} \cdot g_6^{[3]}\right) \tag{13}$$

## 4 CONCLUSION

Let be a spatial curve which is given at the basic coordinate $S_Z = S_{-1}$ by the parametric equations

$$\begin{aligned} x &= x_z = x_{-1} = \varphi(t), \\ y &= y_Z = y_{-1} = \psi(t), \\ z &= z_Z = z_{-1} = \xi(t), t \in < t_0, t_1 > \end{aligned} \tag{14}$$

with the parameter   chosen so that the motion along this curve has in advance given velocity, i.e. there is given the velocity magnitude

$$v(t) = \sqrt{\varphi'^2(t) + \psi'^2(t) + \xi'^2(t)}.$$

Let V be the given point in the space which in the basic coordinate system has the coordinates $V = [v_1, v_2, v_3]^T$. Then the generalized conical surface (see Fig. 6) given by the curve k and the point V has the parametric equations (with respect to the basic coordinate system).

$$\begin{aligned} x &= \varphi(t) + (v_1 - \varphi(t))s, \\ y &= \psi(t) + (v_2 - \psi(t))s, \\ z &= \xi(t) + (v_3 - \xi(t))s, t \in < t_0, t_1 >, s \in < 0, 1 >. \end{aligned} \tag{15}$$

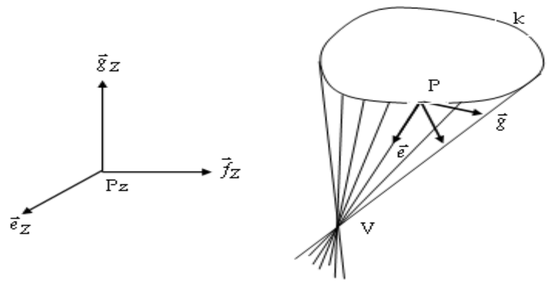


**Figure 6.** Generalized conical surface

We shall require that the motion of the tool is so that its working point is at the time t at the related point of the curve k, i.e. [Modrak 2002, Pasko 2008]

$$P^{[-1]} = [\varphi(t), \psi(t), \xi(t)]^T$$

and the tool is oriented so that $\vec{e} \uparrow\uparrow \overrightarrow{PV}$, the vector $\vec{f}$ is orthogonal to the considered conical surface, and the vector $\vec{g}$ completes the system as the orthonormal and anticlockwise oriented one, i.e.

$$
\begin{aligned}
&e^{[-1]} = \frac{(v_1 - \varphi(t), v_2 - \psi(t), v_3 - \xi(t))^T}{\sqrt{(v_1 - \varphi(t))^2 + (v_2 - \psi(t))^2 + (v_3 - \xi(t))^2}},\\
&u^{[-1]} = (\varphi'(t), \psi'(t), \xi'(t))^T,\\
&v^{[-1]} = e^{[-1]} \times u^{[-1]}, \qquad (16)\\
&f^{[-1]} = \frac{v^{[-1]}}{\left|v^{[-1]}\right|},\\
&g^{[-1]} = e^{[-1]} \times f^{[-1]}.
\end{aligned}
$$

Hence, at any time instant t there is the transform matrix determined [Modrak 2002]

$$
\overline{T}(t) = \begin{bmatrix} e^{[-1]} & f^{[-1]} & g^{[-1]} & p^{[-1]} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

and the values of the generalized coordinates can be determined by the procedure from the previous section.

In many cases the designers have to know the boundaries of the mechanism motion within all moving members of the mechanism. The motion envelope determines the industrial robot's manipulation area for specific operations thus enabling to create a model of a robotized workplace so that if can suit the ideas of a future operation. The envelope of the industrial robot's motion with the end member motion along a conical area can be seen in Fig. 7.

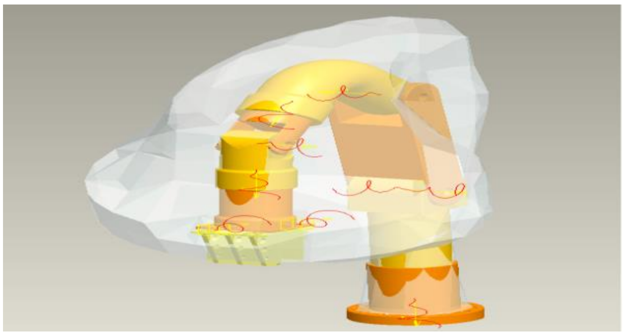

**Figure 7.** The motion envelope


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**CONTACTS:**


Assoc. Prof. M.Sc. Stefan Gaspar, PhD.
Prof. M.Sc. Jan Pasko, PhD.

Technical Universtity of Kosice
Faculty of Manufacturing Technologies with a seat in Presov
Department of Design Technical Systems
Bayerova 1, 080 01 Presov, Slovak Republic
Tel.: +421 51 772 2604
stefan.gaspar@tuke.sk, jan.pasko@tuke.sk
www.tuke.sk